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Auxiliary fields in the geometrical relativistic particle dynamics

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Abstract
We describe how to construct the dynamics of relativistic particles, following either timelike or null curves, by means of an auxiliary variables method instead of the standard theory of deformations for curves. There are interesting physical particle models governed by actions that involve higher order derivatives of the embedding functions of the worldline. We point out that the mechanical content of such models can be extracted wisely from a lower order action, which can be performed by implementing in the action a finite number of constraints that involve the geometrical relationship structures inherent to a curve and by using a covariant formalism. We emphasize our approach for null curves. For such systems, the natural time parameter is a pseudo-arclength whose properties resemble those of the standard proper time. We illustrate the formalism by applying it to some models for relativistic particles.

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1. Introduction

The first geometrical models for rigid particles result as a byproduct of the point-like versions for highly dimensional models that involve the extrinsic curvature of the worldvolume swept out by relativistic strings or branes [1]. Thenceforth, the interest in this sort of particle models has grown by leaps and bounds because one can find potential applications both in particle physics and mathematics. For instance, they can describe spinning particles, whether massive or massless, defined on timelike trajectories [2, 3], and when the model is linear in the geodesic curvature it turns out to be related to a massless particle with W_3 gauge symmetry [4]. The story did not end there. Recently, Nersessian and Ramos proposed certain models for massive particles associated with null curves [5, 6]. Immediately, considerable effort has been devoted by other authors in the understanding of the geometry of these models as well as its applications [7–9]. However, a key drawback for all these models resides in their higher order derivative nature, as a consequence physicists have been reluctant to consider their study due to technical difficulties such as the increasing of the degrees of freedom and the equations of motion being at least of fourth order in derivatives of the fields which do not appear to be tractable. Certainly this unpleasant fact appears to be a great difficulty but these models have the advantage to encode the spin content of the particles in the geometry of the worldlines. The standard way to study these sort of particles is through the theory of deformations sheltered by a Frenet–Serret (FS) basis adapted to the worldline. Unfortunately, this gives rise to lengthy and annoying computations due to the above-mentioned non-trivial higher order derivative property inherent to rigid particles [10, 11]. In this paper we aim to study a powerful tool for the worldline geometry, either timelike or lightlike, namely, the conserved linear momentum whose existence is a simple consequence of the Noether theorem. A striking property of the stress tensor for particles or extended objects is that its conservation in time yields not only the equations of motion but also the intrinsic geometrical properties for every model under consideration [12]. To overcome the majority of the typical technical obstacles in the obtaining of the rigid particle dynamics, we appeal to an auxiliary variables method that was originally introduced for the study of general surfaces and applied to describe fluid membranes [13]. Even though most of the progress in the study of particle models has been made in the spirit of the standard theory of deformations, the conserved linear momentum has not been exploited completely in this context. Therefore, we provide an alternative way to analyse point particle models by means of an easy obtaining of the conserved linear momentum. The main idea behind the work is to replace the original action by one equivalent depending on lower order derivatives evading in this way the standard theory of deformations. We assure that this approach simplifies the dynamical point particle description.

The outline of the paper is as follows. In section 2 we begin with a glimpse of the worldline Frenet–Serret geometry describing both, timelike and lightlike, particle trajectories. This brief section will serve mainly to explain our notation and the basic facts to be used in this paper. In section 3 we apply an auxiliary variables method to obtain the conserved linear momentum associated with a local geometrical action depending on the geodesic curvature and the torsion. We emphasize our approach for the case of null curves since the existing point particle models with this geometry are not widely known. We conclude in section 4 by mentioning some comments. We have tried throughout the paper to follow an index-free notation in order to avoid a cumbersome notation. Definitions of constructed deformations which are helpful to understand the geometrical nature of a particle worldline and important identities of the theory of deformations for curves have been collected in appendix A. To complement our approach in the null case, we obtain the Casimir invariants associated with the Poincaré symmetry which is the subject of appendix B. In our context, these are useful to integrate the equations of motion.

2. Worldline geometry

2.1. Timelike curves

Consider a relativistic particle whose timelike worldline can be described by the embedding $x^\mu = X^\mu(\xi)$, where x^μ are local coordinates in Minkowski spacetime with the metric $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, \ldots, +1)$ and $(\mu, \nu = 0, 1, \ldots, N-1)$, ξ is an arbitrary parameter and X^μ are the embedding functions. The vector tangent to the worldline is given by $\dot{X}^\mu = \mathrm{d}X^\mu/\mathrm{d}\xi$

such that the one-dimensional metric along the curve is $\gamma = \eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu \equiv \dot{X}\cdot\dot{X}$. We assume that for timelike curves $\dot{X}^2 < 0$ is satisfied. The infinitesimal arclength for the worldline is given by

$$\mathrm{d}\tau = (-\dot{X}\cdot\dot{X})^{1/2}\mathrm{d}\xi. \tag{1}$$

This arclength is invariant under reparametrizations of the worldline. We introduce $N-1$ normal vectors to the worldline, denoted by $n^\mu{}_i \quad (i = 1, 2, \ldots, N-1)$. These are defined implicitly by $n^i\cdot\dot{X} = 0$ and normalized as $n_i\cdot n_j = \delta_{ij}$.

Though we may choose to label points along the curve arbitrarily, the most convenient approach to study the dynamics for relativistic particles is to let the parameter to be the arclength along the worldline. We will denote with a prime differentiation with respect to τ. Therefore, we introduce the orthonormal basis $\{X', \eta_i\}$ which satisfy $X'\cdot X' = -1$, $X'\cdot\eta_i = 0$ and $\eta_i\cdot\eta_j = \delta_{ij}$. This basis obeys the following *N*-dimensional FS equations [10, 14]:

$$\begin{aligned}
&X'' = k_1\eta_1,\\
&\eta_1' = k_1X' - k_2\eta_2,\\
&\eta_2' = k_2\eta_1 - k_3\eta_3,\\
&\ldots\quad\ldots\\
&\eta_{N-2}' = k_{N-2}\eta_{N-3} - k_{N-1}\eta_{N-1},\\
&\eta_{N-1}' = k_{N-1}\eta_{N-2},
\end{aligned} \tag{2}$$

where k_i stands for the independent *i*th FS curvature and $k := k_1$ is known as the *geodesic curvature*. Note that from the FS equations (2) we can express the geodesic curvature as

$$k_1 = -X'\cdot\eta_1'. \tag{3}$$

Also note that the geodesic curvature is given in terms of the second-order derivatives of the embedding functions, $k_1 = \sqrt{X''\cdot X''}$.

2.2. Lightlike curves

We now turn to consider a null curve, for the sake of simplicity, in a 3 + 1 ambient Minkowski spacetime with the metric $\eta_{\mu\nu}$ described by the embedding $x^\mu = \mathsf{X}^\mu(\rho)$, where x^μ are local coordinates in the background spacetime, ρ is an arbitrary parameter and X^μ are the embedding functions ($\mu = 0, 1, 2, 3$). Hereafter, in order to compare with respect to the timelike case (see, for instance, (1) and (4)), we consider the signature of $\eta_{\mu\nu}$ to be $(+, -, -, -)$. With this convention timelike vectors have a positive norm. The tangent vector to the curve is given by $\dot{\mathsf{X}}^\mu = \mathrm{d}\mathsf{X}^\mu/\mathrm{d}\rho$. It satisfies that $\dot{\mathsf{X}}\cdot\dot{\mathsf{X}} = 0$ since the curve lies on the light cone so the arclength vanishes. This null condition on the tangent vectors shatters our accustomed vision of the worldline geometry which leads us to promote $\Upsilon = \ddot{\mathsf{X}}\cdot\ddot{\mathsf{X}}$ as the corresponding worldline metric [5]. This new point of view necessarily forces the introduction of a new parameter called pseudo-arclength which becomes fruitful to normalize the derivative of the lightlike tangent vector [5, 15]. The infinitesimal pseudo-arclength for a null curve is given by

$$\mathrm{d}\sigma = (-\ddot{\mathsf{X}}\cdot\ddot{\mathsf{X}})^{1/4}\mathrm{d}\rho. \tag{4}$$

This pseudo-arclength is invariant under reparametrizations of the curve. We shall use again a prime to denote the derivation with respect to σ. To analyse the geometry of null curves, it is desirable to adapt a FS frame constructed in a similar way as in the timelike case [5, 7, 15]. In such approach we consider a basis adapted to null curves spanned by $\{e_+, e_1, e_-, e_2\}$, where

e_+ and e_- are lightlike whilst e_1 and e_2 are spacelike. The null FS basis has the structure

$$\begin{aligned}
&e_+ = \mathsf{X}', \\
&e_+^2 = e_-^2 = 0, \\
&e_\pm \cdot e_1 = e_\pm \cdot e_2 = e_1 \cdot e_2 = 0, \\
&e_+ \cdot e_- = -e_1 \cdot e_1 = -e_2 \cdot e_2 = 1.
\end{aligned}$$

This basis obeys the following four-dimensional FS equations [5, 7, 15]:

$$e_+' = e_1, \tag{5a}$$

$$e_1' = \kappa_1 e_+ + e_-, \tag{5b}$$

$$e_-' = \kappa_1 e_1 + \kappa_2 e_2, \tag{5c}$$

$$e_2' = \kappa_2 e_+, \tag{5d}$$

where κ_1 and κ_2 are independent curvature functions of the null curve similarly as in the timelike case. Occasionally, κ_1 is known as the torsion due to its dependence of the third-order derivatives of the field variables. Note that the torsion can be expressed in several forms. For our purposes below, one convenient way is

$$\kappa_1 = \tfrac{1}{2} e_+'' \cdot e_+''. \tag{6}$$

It is worth noting that κ_1 is given in terms of the third-order derivatives of the field variables, $2\kappa_1 = (\mathsf{X}''' \cdot \mathsf{X}''')$.

3. FS dynamics

3.1. Timelike case

We assume that the dynamics of a rigid particle is specified by an action invariant under reparametrizations of the timelike worldline of the form

$$S_0[X] = \int \mathrm{d}\tau\, L(k_1), \tag{7}$$

where L is a scalar under reparametrizations. It is usual that under an infinitesimal deformation of the embedding $X \to X + \delta X$ the response of the functional (7) casts out the equations of motion and the Noether charges [10, 12]. To accommodate an auxiliary variables method describing rigid particles, we follow the seminal work given in [13]. We would like to distribute this deformation among the parametrization, the FS basis and k_1. That is why we consider them as new independent variables. To promote them as intermediate auxiliary variables, it is necessary to implement constraints involving their definitions smeared out with Lagrange multipliers.

Thus, we now construct a new functional action $S[k_1, \eta_1, X', X, f, \lambda^1, \lambda^{11}, \lambda]$ of the form

$$\begin{aligned}
S = S_0[X, k_1] &+ \int \mathrm{d}\tau\, f \cdot \left(X' - \frac{\mathrm{d}X}{\mathrm{d}\tau} \right) + \int \mathrm{d}\tau [\lambda^1 (X' \cdot \eta_1) + \lambda^{11} (\eta_1 \cdot \eta_1 - 1)] \\
&+ \int \mathrm{d}\tau\, \lambda (k_1 + X' \cdot \eta_1').
\end{aligned} \tag{8}$$

This is a suitable departure point which provides both geometrical and physical insights into the mechanical systems described by (7) by means of a conserved linear momentum.

The Euler–Lagrange (EL) derivative for X is such that in the extremum condition it shows a conservation law,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\{f^\mu + [L + \lambda k_1 + (f\cdot X')]X'^\mu\} = 0, \tag{9}$$

where we have employed identities (A.3) and expression (3).

The EL derivative associated with X' exhibits the geometrical form of f^μ in terms of the Lagrange multipliers and the FS basis

$$f = -\lambda k_1 X' - \lambda^1\eta_1 + \lambda k_2\eta_2, \tag{10}$$

where we have used expressions (2). As a result we obtain $(f\cdot X') = \lambda k_1$. Correspondingly, the EL derivative for η_1 and by exploiting the FS equations (2), we have

$$\lambda^1 = \lambda', \qquad 2\lambda^{11} = \lambda k_1. \tag{11}$$

Finally, the EL derivative for k_1 yields

$$\lambda = -L^* \tag{12}$$

where we have introduced the notation $L^* = \mathrm{d}L/\mathrm{d}k_1$. Thus, we can identify the Lagrange multipliers (11) as $\lambda^1 = -L^{*\prime}$ and $2\lambda^{11} = -k_1L^*$.

Hence, putting all of these results together in the conservation law (9), we therefore get

$$\frac{\mathrm{d}}{\mathrm{d}\tau}[(L - L^*k_1)X' + L^{*\prime}\eta_1 - L^*k_2\eta_2] = 0, \tag{13}$$

which allows us to identify the conserved linear momentum, written in terms of the FS basis,

$$p = (L - L^*k_1)X' + L^{*\prime}\eta_1 - L^*k_2\eta_2. \tag{14}$$

This is nothing but the linear momentum associated with the Noether charge specialized to a constant infinitesimal translation $\delta X^\mu = \epsilon^\mu$ [10]. Further, momentum (14) is in accordance with the momentum conjugated to the embedding variables in an Ostrogradski Hamiltonian approach for action (7) [16].

The FS projections of the total derivative (13) permit us to deduce the mechanical and geometrical properties of the generic action (7). The projection of (13) along η_3 implies the vanishing of k_3 thereby the motion is performed in $2+1$ dimensions. Similarly, the projection of (13) along η_2 leads to $(L^*)^2k_2 = \text{const}$, which can be interpreted as a conservation of the spin of the particle [10]. To finish the tangential projection casts out the equations of motion, namely, $L^{*\prime\prime} + (L - L^*k_1)k_1 - L^*k_2^2 = 0$. These properties as well as the Poincaré invariants have been well discussed in [10, 11].

In closing this subsection, we apply the formalism developed above to the linear correction to the free relativistic particle, $L = -m + \alpha k_1$, where m and α are constants. Obviously we have $L^* = \alpha$. The corresponding linear momentum is given by $p = -mX' - \alpha k_2\eta_2$ and the equation of motion results $\alpha m k_1 + \text{const} = 0$.

3.2. Lightlike case

Now, we shall consider actions for null curves that are invariant under reparametrizations of the form

$$S_0[\mathsf{X}] = \int \mathrm{d}\sigma\, L(\kappa_1), \tag{15}$$

where L is invariant under worldline reparametrizations. An auxiliary variables method will distribute the deformation $\mathsf{X} \to \mathsf{X} + \delta\mathsf{X}$ among X itself, e_+ and κ_1 considering all of them as new independent variables. Once again, bearing in mind the necessity to promote them as auxiliary variables, we need to implement them through their definitions and structure properties smeared with appropriated Lagrange multipliers.

Following the timelike case, we now construct the functional $S[\kappa_1, e_+, \mathsf{X}, \mathsf{f}, \Lambda_{++}, \Lambda]$ written as

$$S = S_0[\mathsf{X}, \kappa_1] + \int \mathrm{d}\sigma\, \mathsf{f}\cdot\left(e_+ - \frac{\mathrm{d}}{\mathrm{d}\sigma}\mathsf{X}\right) + \int \mathrm{d}\sigma\, \Lambda_{++} e_+^2 + \int \mathrm{d}\sigma\, \Lambda\left(\kappa_1 - \frac{1}{2} e_+'' \cdot e_+''\right). \tag{16}$$

A direct computation of the EL derivative for X shows that in the extremum condition we have

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\left\{\mathsf{f}^\mu - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\sigma}\left[(L + 4\Lambda\kappa_1 + (\mathsf{f}\cdot e_+))\, e_1^\mu\right]\right\} = 0, \tag{17}$$

where we have employed identities (A.6) and expression (6). The EL derivative for e_+ allows us to write f^μ in terms of the null FS frame as

$$\mathsf{f} = (\Lambda e_1')'' - 2\Lambda_{++} e_+. \tag{18}$$

By making use of the null FS equations, we obtain $(\mathsf{f}\cdot e_+) = 2\Lambda\kappa_1 + \Lambda''$. Finally, we compute the EL derivative with respect to κ_1,

$$\Lambda = -L^*, \tag{19}$$

where we have used one more time $*$ to denote the derivative with respect to κ_1, i.e., $L^* = \mathrm{d}L/\mathrm{d}\kappa_1$. We are ready to insert the information into the conservation law (17). With the previous results we obtain $\mathsf{f} = -(L^* e_1')'' - 2\Lambda_{++} e_+$, such that the conservation law (17) reads

$$\mathsf{E} = \frac{\mathrm{d}}{\mathrm{d}\sigma}\left\{2\Lambda_{++} e_+ + (L^* e_1')'' + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\sigma}[(L - 6L^*\kappa_1 - L^{*\prime\prime}) e_1]\right\} = 0, \tag{20}$$

which helps us to identify the corresponding linear momentum, $\mathsf{p} = \mathsf{p}_+ e_+ + \mathsf{p}_- e_- + \mathsf{p}_1 e_1 + \mathsf{p}_2 e_2$, in the null FS frame. If the conservation law (20) is expressed as $\mathsf{E}^\mu = \mathsf{p}^{\mu'} = 0$, it is straightforward to obtain the conditions that the momentum components must satisfy,

$$\mathsf{p}_+' + \mathsf{p}_1\kappa_1 + \mathsf{p}_2\kappa_2 = 0, \qquad \mathsf{p}_-\kappa_2 + \mathsf{p}_2' = 0, \tag{21}$$

and

$$\mathsf{p}_1 + \mathsf{p}_-' = 0, \qquad \mathsf{p}_+ + \mathsf{p}_1' + \mathsf{p}_-\kappa_1 = 0, \tag{22}$$

where the FS equations in the null frame have been exploited. Equations (21) are in fact the equations of motion of the particles governed by (15), whereas (22) are simple geometrical identities. The momentum acquires the form

$$\mathsf{p} = (\mathsf{p}_-'' - \mathsf{p}_-\kappa_1) e_+ + \mathsf{p}_- e_- - \mathsf{p}_-' e_1 + \mathsf{p}_2 e_2.$$

We remark at this point that it is not necessary to know the form that Λ_{++} must hold.

A straightforward computation in (20) leads us to identify the independent components of the momentum in terms of the worldline curvatures, $\mathsf{p}_- = (L - 2L^*\kappa_1 + L^{*\prime\prime})/2$ and $\mathsf{p}_2 = 2L^{*\prime}\kappa_2 + L^*\kappa_2'$. Thus, we can write the linear momentum in the null FS frame:

$$\begin{aligned}\mathsf{p} = -\tfrac{1}{2}&[(L - 2L^*\kappa_1 + L^{*\prime\prime})\kappa_1 - (L - 2L^*\kappa_1 + L^{*\prime\prime})'']e_+ \\ &+ \tfrac{1}{2}(L - 2L^*\kappa_1 + L^{*\prime\prime}) e_- - \tfrac{1}{2}(L - 2L^*\kappa_1 + L^{*\prime\prime})' e_1 \\ &+ (2L^{*\prime}\kappa_2 + L^*\kappa_2') e_2.\end{aligned} \tag{23}$$

This is the general expression for the momentum associated with (15). It is worth pointing out that momentum (23) is completely determined by two independent components p_- and p_2 in the four-dimensional case. In fact, this is bequeathed from the theory of deformations where in order to preserve null curves in the variational procedure, two independent normal variations are necessary [15].

To project the conservation law (20) into the null FS frame is equivalent to express equations (21) and (22) in terms of the independent components of momentum (23). The first equation of (22) is

$$\mathsf{E}_- = L' - L^*\kappa_1' = 0. \tag{24}$$

This is merely an identity based in the chain rule from ordinary calculus. The second equation of (22), E_1, is just a null identity. Now, the second equation of (21) is written as

$$\mathsf{E}_2 = (L - 2L^*\kappa_1 + L^{*\prime\prime})\kappa_2 + 2[(L^{*2}\kappa_2)'/L^*]' = 0. \tag{25}$$

This equation determines κ_2 in terms of κ_1. Finally, the first equation of (21) results

$$\begin{aligned}\mathsf{E}_+ = (L - 2L^*\kappa_1 + L^{*\prime\prime})''' - 2(L - 2L^*\kappa_1 + L^{*\prime\prime})'\kappa_1 \\ - (L - 2L^*\kappa_1 + L^{*\prime\prime})\kappa_1' + 2[(L^{*2}\kappa_2)'/L^*]\kappa_2 = 0.\end{aligned} \tag{26}$$

Expressions (25) and (26) determine the equations of motion governing the dynamics of particles described by action (15) which do not appear to be tractable in general. In fact, in the most simple cases they are two coupled differential equations whose solutions are null helices [7, 17]. There, the equations of motion can be integrated and expressed in terms of the mass and the spin of the particle [15]. We must remark that in a $2+1$ ambient Minkowski spacetime, besides $\kappa_2 = 0$, the momentum component p_2 disappears and the only equation of motion is $(\mathsf{p}_-'' - \mathsf{p}_-\kappa_1)' - \mathsf{p}_-'\kappa_1 = 0$. This latter equation also appears to be intractable in general but surprisingly we find that for an arbitrary Lagrangian L it is possible to reduce the order of the equation of motion. We show briefly how this comes about. In general, the equations of motion are equivalent to the associated constants of motion given by the first and second Casimir invariants (see, for example, [9] for a proof of this statement). By putting expressions (B.2) and (B.7) together, we find that

$$L^{*\prime}(\mathsf{p}_-^2)' - L^*[(\mathsf{p}_-')^2 + M^2] + (L - L^{*\prime\prime})\mathsf{p}_-^2 + 2S\mathsf{p}_- = 0. \tag{27}$$

The immediate implication of this ODE in κ_1 is that it is equivalent to the original equation of motion besides reduced in the order. In a $3+1$ ambient spacetime, the integration for an arbitrary L can be treated along the same lines but the computation is rather involved.

We survey the application of the formalism by considering first a model for particles, in a $3+1$ ambient spacetime, given by a correction to the pseudo-arclength parameter Lagrangian, $L = 2(\alpha + \beta\kappa_1)$, where α and β are constants. Obviously, $L^* = 2\beta$. The associated linear momentum is given by

$$\mathsf{p} = -[\beta\kappa_1'' + (\alpha - \beta\kappa_1)\kappa_1]e_+ + (\alpha - \beta\kappa_1)e_- + \beta\kappa_1' e_1 + 2\beta\kappa_2' e_2. \tag{28}$$

Hence, from (25) and (26), the equations of motion are

$$\beta\kappa_1''' - \frac{3\beta}{2}\kappa_1'^2 - \beta\kappa_2'^2 + \alpha\kappa_1' = 0, \tag{29}$$

$$2\beta\kappa_2'' - \beta\kappa_1\kappa_2 + \alpha\kappa_2 = 0, \tag{30}$$

which can be integrated and expressed in terms of the Casimir invariants. Recently, these equations of motion have been extensively studied in [7, 17].

Another example more complicated than the previous one can be found in a $2+1$ ambient spacetime. Consider $L = 2\left(\alpha + \beta\kappa_1^2\right)$ with α and β being constants. This model resembles to the 1 + 1 timelike effective model for a relativistic kink in the field of a soliton [18]. The corresponding linear momentum is

$$\mathsf{p} = \{\beta(-3\kappa_1^2 + 2\kappa_1'')'' - [\alpha + \beta(-3\kappa_1^2 + 2\kappa_1'')]\kappa_1\}e_+ \\ + [\alpha + \beta(-3\kappa_1^2 + 2\kappa_1'')]e_- - \beta(-3\kappa_1^2 + 2\kappa_1'')'e_1. \tag{31}$$

From (26) we obtain the equation of motion

$$\{\beta(-3\kappa_1^2 + 2\kappa_1'')'' - 2[\alpha + \beta(-3\kappa_1^2 + 2\kappa_1'')]\kappa_1\}' + \alpha\kappa_1' + \beta(-3\kappa_1^2 + 2\kappa_1'')\kappa_1' = 0, \tag{32}$$

which can be integrated immediately to give a fourth-order ODE in κ_1,

$$\kappa_1^{(4)} - 5\kappa_1\kappa_1'' - \tfrac{5}{2}(\kappa_1')^2 + \tfrac{5}{2}\kappa_1^3 - \gamma\kappa_1 - \lambda_{(3)} = 0, \tag{33}$$

where $\gamma = \alpha/2\beta$ and $\lambda_{(3)}$ is an integration constant which, in principle, can be written in terms of the Casimir invariants. One can go further on the integration of equation (32) if we appeal to expression (27). The original equation of motion is equivalent to

$$2\beta\kappa_1\{[\alpha + \beta(-3\kappa_1^2 + 2\kappa_1'')]'^2 + M^2\} - 2\beta\kappa_1'\{[\alpha + \beta(-3\kappa_1^2 + 2\kappa_1'')]^2\}' \\ - (\alpha + \beta\kappa_1^2 - 2\beta\kappa_1'')[\alpha + \beta(-3\kappa_1^2 + 2\kappa_1'')]^2 + S[\alpha + \beta(-3\kappa_1^2 + 2\kappa_1'')] = 0, \tag{34}$$

where M^2 is the first Casimir invariant given by

$$M^2 = \{[\alpha + \beta(\kappa_1^2 - 2\beta\kappa_1'')]^2\}'' - 3\beta^2[(-3\kappa_1^2 + 2\kappa_1'')']^2 - 2[\alpha + \beta(\kappa_1^2 - 2\beta\kappa_1'')]^2\kappa_1, \tag{35}$$

and S is the associated second Casimir invariant which becomes

$$S = -[\alpha + \beta(\kappa_1^2 - 2\kappa_1'')][\alpha + \beta(-3\kappa_1^2 + 2\kappa_1'')] - 4\beta^2\kappa_1'(-3\kappa_1^2 + 2\kappa_1'')' \\ + 4\beta\kappa_1[-\alpha\kappa_1 + \beta(-3\kappa_1^2 + 2\kappa_1'')'' - \beta\kappa_1(-3\kappa_1^2 + 2\kappa_1'')]. \tag{36}$$

Despite we have enormously reduced the order of the original equation of motion, the equivalent equation (34) turns out to be complicated as opposed to equation (33). The main benefit of (33) resides in its simplicity. In fact, alike to (33) is the resulting equation of motion for a particle described by a model linear in κ_1 in a 3 + 1 ambient spacetime [17].

4. Concluding remarks

In this paper, we have analysed worldline theories by obtaining the associated conserved linear momentum. This has been done by means of an auxiliary variables method. The main advantage of the method is based in the reduction of the higher order derivative nature of the fields, obtaining considerable simplification in the variational procedure and avoiding awkward computations. We have tailored this auxiliary variables method to the FS frame of each curve, either timelike or lightlike. Based on the Poincaré and reparametrization invariance of the action, the conservation of the momentum leads us to the full mechanical content of the worldline theories. Equations (13) and (20) provide the dynamics for arbitrary Lagrangians $L(k_1)$ and $L(\kappa_1)$, when they are implemented by the FS frame projections. We showed that the auxiliary variables method is in fact a powerful alternative to study embedded theories. Although originally it was implemented to study general surfaces characterized by their extrinsic geometry, like the lipid membranes [13], its application is immediate to relativistic brane models under interaction with other fields. The complete integrability of the equations of motion faces several technical difficulties when one tries to integrate them for a general Lagrangian $L(\kappa_1)$. It seems to be intractable in general. We explored some examples to see our machinery at work. For the simplest cases, like a constant and linear in κ_1, the integrability is a well-known fact. Nevertheless, we have shown the existence of other model, quadratic in κ_1, in a 2 + 1 ambient spacetime where we have also obtained integrability. Work along this issue is in progress.

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Appendix A

A.1. Arclength infinitesimal geometry

In this appendix, we express the main variations necessary to develop the dynamics from a general action functional using the proper time as the parameter to describe the curve [10, 11, 19].

For an infinitesimal deformation of a timelike worldline, $X^\mu(\xi) \to X^\mu(\xi) + \delta X^\mu(\xi)$, we can decompose the deformation with respect to the FS basis as

$$\delta X = \Phi X' + \Psi_i \eta_i. \tag{A.1}$$

The tangential projection can be identified with reparametrizations of the worldline. The tangential deformation of the proper time (1) is $\delta_\parallel \, \mathrm{d}\tau = \Phi' \, \mathrm{d}\tau$. It follows straightforwardly that

$$\delta \mathrm{d}\tau = (\Phi' + k_1 \Psi_1) \, \mathrm{d}\tau = -\left(X' \cdot \frac{\mathrm{d}}{\mathrm{d}\tau} \delta X \right) \mathrm{d}\tau, \tag{A.2}$$

$$\left[\delta, \frac{\mathrm{d}}{\mathrm{d}\tau} \right] = -(\Phi' + k_1 \Psi_1) \frac{\mathrm{d}}{\mathrm{d}\tau} = \left(X' \cdot \frac{\mathrm{d}}{\mathrm{d}\tau} \delta X \right) \frac{\mathrm{d}}{\mathrm{d}\tau}, \tag{A.3}$$

which help us to recognize the dependence on the parametrization of the proper time and the FS basis through the prime derivatives and show that the operations δ and $\mathrm{d}/\mathrm{d}\tau$ do not commute [10].

A.2. Pseudo-arclength infinitesimal geometry

For an infinitesimal deformation of a null curve, $\mathsf{X}^\mu(\rho) \to \mathsf{X}^\mu(\rho) + \delta \mathsf{X}^\mu(\rho)$, we can express the deformation with respect to the null FS frame as

$$\delta \mathsf{X} = \epsilon_+ e_+ + \epsilon_- e_- + \epsilon_1 e_1 + \epsilon_2 e_2. \tag{A.4}$$

Similarly as in the timelike case, the tangential projection can be identified with reparametrizations of the null curve such that the tangential infinitesimal deformation of the pseudo-arclength (4) is given by $\delta_\parallel \, \mathrm{d}\sigma = \epsilon'_+ \, \mathrm{d}\sigma$.

To preserve the null character of the curve, the condition $\delta(\dot{\mathsf{X}} \cdot \dot{\mathsf{X}}) = 0$ leads us to the constraint

$$\epsilon_1 + \epsilon'_- = 0, \tag{A.5}$$

on the components of the deformation. Explicitly, this condition is equivalent to $e_+ \cdot \frac{\mathrm{d}}{\mathrm{d}\sigma} \delta \mathsf{X} = 0$.

In the null FS frame, one can show analogously the identities

$$\delta \, \mathrm{d}\sigma = \frac{1}{2} (2\epsilon'_+ - \epsilon'''_- + \kappa'_1 \epsilon_- + \kappa_2 \epsilon_2) \, \mathrm{d}\sigma = -\frac{1}{2} \left(e_1 \cdot \frac{\mathrm{d}^2}{\mathrm{d}\sigma^2} \delta \mathsf{X} \right) \mathrm{d}\sigma,$$

$$\left[\delta, \frac{\mathrm{d}}{\mathrm{d}\sigma}\right] = -\frac{1}{2}(2\epsilon_+' - \epsilon_-''' + \kappa_1'\epsilon_- + \kappa_2\epsilon_2)\frac{\mathrm{d}}{\mathrm{d}\sigma} = \frac{1}{2}\left(e_1 \cdot \frac{\mathrm{d}^2}{\mathrm{d}\sigma^2}\delta\mathsf{X}\right)\frac{\mathrm{d}}{\mathrm{d}\sigma}.$$

Similarly, as in the timelike case, the operations δ and $\mathrm{d}/\mathrm{d}\sigma$ do not commute [15].

Appendix B. Noether invariants in the lightlike case

The writing of the first variation of action (15) in the form

$$\delta S = \int \mathrm{d}\sigma\, \mathsf{E}\cdot\delta\mathsf{X} + \int \mathrm{d}\sigma\, \mathsf{Q}',$$

where E^μ stands for the EL derivative associated with X, entails the identification of the associated Noether charge Q given by [10]

$$\mathsf{Q} = \mathsf{p}\cdot\delta\mathsf{X} - \frac{1}{2}[(L - L^{*\prime\prime} + L^*\kappa_1)e_1 + 2L^{*\prime}e_- + 2L^*\kappa_2 e_2]\cdot\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta\mathsf{X} + \frac{1}{2}[(2L^*\kappa_1 - L^{*\prime})e_1 + 3L^*e_-]\cdot\frac{\mathrm{d}^2}{\mathrm{d}\sigma^2}\delta\mathsf{X} + \frac{1}{2}L^*e_1\cdot\frac{\mathrm{d}^3}{\mathrm{d}\sigma^3}\delta\mathsf{X}. \tag{B.1}$$

It is clear that if the deformation $\delta\mathsf{X}^\mu$ is a constant infinitesimal deformation, $\delta\mathsf{X}^\mu = \varepsilon^\mu$, and assuming $\mathsf{Q} = \varepsilon^\mu\mathsf{p}_\mu$, we are able to recuperate the expression for linear momentum (23). The first Casimir invariant of the Poincaré group, $M^2 = \mathsf{p}^2$, results

$$\begin{aligned} M^2 &= (\mathsf{p}_-^2)'' - 3(\mathsf{p}_-')^2 - 2\mathsf{p}_-^2\kappa_1 - \mathsf{p}_2^2 \\ &= \frac{1}{4}[(L - 2L^*\kappa_1 + L^{*\prime\prime})^2]'' - \frac{3}{4}[(L - 2L^*\kappa_1 + L^{*\prime\prime})']^2 \\ &\quad - \frac{1}{2}(L - 2L^*\kappa_1 + L^{*\prime\prime})^2\kappa_1 - [(L^{*2}\kappa_2)'/L^*]^2. \end{aligned} \tag{B.2}$$

On the other hand, by specializing the deformation $\delta\mathsf{X}^\mu$ to Lorentz transformations, $\delta\mathsf{X}^\mu = \omega^\mu{}_\nu\mathsf{X}^\nu$ with $\omega_{\mu\nu} = -\omega_{\nu\mu}$ and assuming $\mathsf{Q} = \omega_{\mu\nu}\mathsf{M}^{\mu\nu}$, we obtain the conserved angular momentum

$$\mathsf{M}^{\mu\nu} = \mathsf{p}^{[\mu}X^{\nu]} + \tfrac{1}{2}(L - L^{*\prime\prime})e_+^{[\mu}e_1^{\nu]} + L^{*\prime}e_+^{[\mu}e_-^{\nu]} + L^*\kappa_2 e_+^{[\mu}e_2^{\nu]} + L^*e_-^{[\mu}e_1^{\nu]}. \tag{B.3}$$

If the particle moves in a 3 + 1 ambient spacetime, to extract the spin content of the particle models governed by action (15), we introduce the Pauli–Lubanski pseudo-vector, $S_\mu = \frac{1}{2\sqrt{|M^2|}}\varepsilon_{\mu\nu\rho\sigma}\mathsf{p}^\nu\mathsf{M}^{\rho\sigma}$, which results

$$S_\mu = \frac{1}{2\sqrt{|M^2|}}\left\{-\left[\mathsf{p}_-'L^*\kappa_2 + \frac{1}{2}\mathsf{p}_2(L - L^{*\prime\prime})\right]e_{+\mu} + (\mathsf{p}_2L^*)e_{-\mu} - (\mathsf{p}_2L^*)'e_{1\mu} - \left[(\mathsf{p}_-\kappa_1 - \mathsf{p}_-'')L^* + \frac{1}{2}\mathsf{p}_-(L - L^{*\prime\prime}) + \mathsf{p}_-'L^{*\prime}\right]e_{2\mu}\right\}, \tag{B.4}$$

where $\varepsilon_{\alpha\beta\rho\sigma}$ is the Levi-Civita tensor density and we have used the following convention $\varepsilon_{\alpha\beta\rho\sigma}e_+^\alpha e_-^\beta e_1^\rho e_2^\sigma = +1$. The second Casimir invariant of the Poincaré group is

$$\begin{aligned} 4|M^2|S^2 &= -\left[(\mathsf{p}_-\kappa_1 - \mathsf{p}_-'')L^* + \tfrac{1}{2}\mathsf{p}_-(L - L^{*\prime\prime}) + \mathsf{p}_-'L^{*\prime}\right]^2 - [(\mathsf{p}_2L^*)']^2 \\ &\quad - 2\left[\mathsf{p}_-'L^*\kappa_2 + \tfrac{1}{2}\mathsf{p}_2(L - L^{*\prime\prime})\right](\mathsf{p}_2L^*). \end{aligned} \tag{B.5}$$

Now, for a 2 + 1 ambient spacetime, the spin content of particles can be extracted from the spin pseudo-vector, $\mathsf{J}_\mu = \varepsilon_{\mu\alpha\beta}M^{\alpha\beta}$, resulting in

$$\mathsf{J}_\mu = \varepsilon_{\mu\alpha\beta}\mathsf{p}^\alpha\mathsf{X}^\beta - \tfrac{1}{2}(L - L^{*\prime\prime})e_{+\mu} - L^{*\prime}e_{1\mu} + L^*e_{-\mu}. \tag{B.6}$$

Thus, we have the second Casimir, $S = \mathsf{J}\cdot\mathsf{p}$, given by

$$S = -\tfrac{1}{2}(L - L^{*\prime\prime})\mathsf{p}_- - L^{*\prime}\mathsf{p}_-^{\prime} + L^*(\mathsf{p}_-^{\prime\prime} - \mathsf{p}_-\kappa_1). \tag{B.7}$$

It should be clear at this point that the Noether invariants of the underlying Poincaré symmetry are expressed in terms of the geometry of the worldline. In addition, the spin content of this sort of particles depends heavily on the worldline curvatures.

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